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CYCLES OF INDECOMPOSABLE MODULES

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The aim of this article is to present some results on cycles of indecomposable modules over artin algebras. We shall show how we may express some properties of modules and algebras in terms of cycles of indecomposable modules.

Throughout the article A will denote a fixed artin algebra over a commutative artin ring R. By an A-module is meant a finitely generated right A-module. We shall denote by mod A the category of all (finitely generated) A-modules, by rad(mod A) the radical of mod A, and by $\mathrm{rad}^{\infty}(\mathrm{mod}\ A)$ the intersection of all powers $\mathrm{rad}^{i}(\mathrm{mod}\ A)$, $i \geq 0$, of rad(mod A). From the existence of Auslander-Reiten sequences in mod A we know that rad(mod A) is generated by the irreducible maps as a left and as a right ideal. It is also known that A is representation-finite if and only if $\mathrm{rad}^{\infty}(\mathrm{mod}\ A) = 0$ (see [KS]). Further, we denote by Γ_A the Auslander-Reiten quiver of A and by DTr the Auslander-Reiten operator on mod A. It is well known that Γ_A describes the quotient category $\mathrm{mod}\ A/\mathrm{rad}^{\infty}(\mathrm{mod}\ A)$.

A <u>cycle</u> in mod A is a sequence

$$(*) \qquad M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n = M_0$$

where all M_i are indecomposable A-modules and all f_i are nonzero and nonisomorphisms. If $n \geqslant 2$, then the cycle (*) is said to be short. Moreover, a cycle (*) is said to be finite if f_i does not belong to $\mathrm{rad}^{\infty}(\mathrm{mod}\ A)$ for any i, $1 \leqslant i \leqslant n$. We know from [R1 ; 2.4] that if A is representation-infinite then mod A always contains a cycle. Moreover, if A is selfinjective (resp. weakly symmetric) then every indecomposable A-module lies on a cycle (resp. short cycle), see [RSS2 ; 2.1].

We shall also need the following notation. For an A-module M, write $A = P \oplus Q$ such that the simple summands of P/rad P are exactly the simple composition factors of M. Then $\mathrm{supp}\ M = \mathrm{End}_A(P)$ is called the support algebra of M. If supp M = A, then M is called sincere. Similarly, for a connected component $\mathcal{C}$ of Γ_A, write $A = P \oplus Q$ such that the simple summands of P/rad P are exactly the simple composition factors of modules in $\mathcal{C}$. Then $\mathrm{supp}\ \mathcal{C} = \mathrm{End}_A(P)$ is called the support algebra of $\mathcal{C}$. For the basic facts concerning the tiltingtheory we refer to [A] and [R1].

1. DIRECTING MODULES

Following [R1] an indecomposable A-module M is called directing if it does not belong to a cycle in mod A. Directing modules have played an important role in the representation theory of algebras: preprojective and preinjective components in general and connecting components of tilted algebras consist entirely of directing modules. Moreover, for representation-finite, finite-dimensional algebras over an algebraically closed field, the classification of indecomposable modules reduces, via coverings, to the classification of directing modules. This is not the case for representation-infinite algebras.

The supports of directing modules are described by the following

THEOREM 1.1. If M is an directing A-module then supp M is a tilted algebra.

Proof. [R1 ;p.376] .

We shall now describe the regular components of Γ_A containing directing modules. Recall that a connected component $\mathcal{C}$ of Γ_A is called regular if $\mathcal{C}$ contains neither a projective nor an injective module.

THEOREM 1.2. Let $\mathcal{C}$ be a connected component of Γ_A consisting entirely of directing modules. Then $\mathcal{C}$ has only finitely many DTr-orbits.

Proof. [SS].

It was proved in [R2] that a hereditary algebra $H = k\Delta$ has a regular tilting module if and only if the quiver Δ of H has at least three vertices and is neither of Euclidean nor of Dynkin type. Moreover, if T is a regular tilting H-module and $B = \mathrm{End}_H(T)$ the associated tilted algebra, then the connecting component of Γ_B is a regular component of type Δ^{op} consisting entirely of directing B-modules.

The following theorem shows that such components exhaust all regular components consisting entirely of directing modules.

THEOREM 1.3. Let $\mathcal{C}$ be a regular connected component of Γ_A consisting entirely of directing modules. Write $A = P \oplus Q$, where supp $\mathcal{C} = \mathrm{End}_A(P)$, and denote by I the ideal in A generated by all images of maps from Q to A. Then

(i) B = supp $\mathcal{C}$ is a tilted algebra of the form $\mathrm{End}_H(T)$ with H a (wild) hereditary algebra and T a regular tilting H-module.

(ii) $B = A/I$.

(iii) $\mathcal{C}$ is the connecting component of Γ_B.

Proof. [SS ; 2.7 and 2.4].

Recently I have proved the following

THEOREM 1.4. Let $\mathcal{C}$ be a regular connected component of Γ_A containing at least one directing module. Then all modules in $\mathcal{C}$ are directing.

Proof. [S2].

Combining the above theorems we obtain the following

COROLLARY 1.5. Γ_A admits at most finitely many connected components containing directing modules.

2. FINITE CYCLES

Following [AS2] the algebra A is called <u>cycle-finite</u> if any cycle in mod A is finite. Obviously, if A is representation-finite then it is cycle-finite (because $\mathrm{rad}^{\infty}(\mathrm{mod}\ A) = 0$). On the other hand, there are many representation-infinite cycle-finite algebras. For example, all representation-infinite tilted algebras of Euclidean type and all tubular algebras are cycle-finite (see [R1]). But for selfinjective algebras we have the following

THEOREM 2.1. Let A be selfinjective. Then A is cycle-finite if and only if A is representation-finite.

Proof. [S3].

Cycle-finite algebras have played an important role in the study of tame finite-dimensional algebras over an algebraically closed field (see [AS2], [AS3], [AS4], [S1]). Assume that R = k is an algebraically closed field. Then A is called <u>tame</u> if, for any dimension d, there exists a finite number of $k[x]$-A-bimodules $M_1, \ldots, M_{n_d}$, where $k[x]$ is the polynomial algebra in one variable, satisfying the following conditions:

(a) For any i, $1 \leq i \leq n_d$, M_i is a free left $k[x]$-module of finite rank.

(b) All but finitely many (up to isomorphism) indecomposable A-modules of dimension d are isomorphic to $k[x]/(x-\lambda) \otimes_{k[x]} M_i$ for some i and some $\lambda \in k$.

We denote by $\mu_A(d)$ the least number of $k[x]$-A-bimodules satisfying the above conditions (a) and (b). Then A is said to be of <u>polynomial growth</u> (resp. <u>domestic</u>) if there is a natural number m such that $\mu_A(d) \leq d^m$ (resp. $\mu_A(d) \leq md$) for all $d \geqslant 1$. Moreover, following [S1], A is said to be <u>strongly</u>

simply connected if any full convex subcategory of A is simply connected in the sense of [AS1].

We have the following characterization of strongly simply connected polynomial growth (resp. domestic) algebras.

THEOREM 2.2. Let R = k be an algebraically closed field and A be strongly simply connected. Then

(i) A is of polynomial growth if and only if A is cycle-finite.

(ii) A is domestic if and only if $\mathrm{rad}^{\infty}(\mathrm{mod}\ A)$ is nilpotent.

Proof. [S1].

Recently I have proved the following general result

THEOREM 2.3. Let R = k be an algebraically closed field and A be cycle-finite. Then A is of polynomial growth.

Proof. [S3].

REMARK 2.4. There are polynomial growth algebras which are not cycle-finite (see [KS ; 1.4]).

3. SHORT CYCLES

The following lemma shows that usually mod A contains many short cycles.

LEMMA 3.1. Let M be an indecomposable A-module such that $\mathrm{Ext}^1_A(M,M) \neq 0$. Then M lies on a short cycle.

Proof. From the Auslander-Reiten formula $\mathrm{Ext}^1_A(M,M) \cong D\overline{\mathrm{Hom}}_A(M,DTrM)$, $\mathrm{Ext}^1_A(M,M) \neq 0$ implies that $\mathrm{Hom}_A(M,DTrM) \neq 0$. Consider an Auslander-Reiten sequence $0 \to DTrM \to E \to M \to 0$. Then there exists an indecomposable direct summand F of E such that $\mathrm{Hom}_A(M,F) \neq 0$. Therefore we have a short cycle $M \to F \to M$.

Short cycles are related with short chains introduced in [AR]. Recall that a chain of two nonzero maps $X \to M \to DTrX$ with X and M indecomposable A-modules is called a short chain and M its middle.

Moreover, we say that a chain $\ldots \longrightarrow C_i \longrightarrow C_{i+1} \longrightarrow \ldots$ with $i \in Z$ of irreducible maps between indecomposable A-modules is A_∞^∞-sectional if $\mathrm{DTr}C_{i+1} \not\cong C_{i-1}$ for all $i \in Z$. Then we have the following

THEOREM 3.2. Let M be an indecomposable A-module. Then

(i) If M is the middle of a short chain then M lies on a short cycle.

(ii) If Γ_A has no A_∞^∞-sectional chains, then M is the middle of a short chain if and only if M lies on a short cycle.

Proof. [RSS1 ; 1.6].

REMARK 3.3. There are many algebras A such that Γ_A has no A_∞^∞-sectional chains. For example, all representation-finite algebras and all hereditary algebras have this property. It would be interesting to know whether the equivalence (ii) is true without assumption on the nonexistence of A_∞^∞-sectional chains in Γ_A.

We have also the following result on regular components consisting of modules which do not lie on short cycles.

THEOREM 3.5. Let R = k be an algebraically closed field and $\mathcal{C}$ a regular connected component of Γ_A having only finitely many DTr-orbits and consisting entirely of modules which do not lie on short cycles. Write $A = P \oplus Q$, where supp $\mathcal{C} = \mathrm{End}_A(P)$, and denote by I the ideal of A generated by all images of maps from Q to A. Then

(i) B = supp $\mathcal{C}$ is a tilted algebra of the form $\mathrm{End}_H(T)$ with H a (wild) hereditary algebra and T a regular tilting H-module.

(ii) B = A/I .

(iii) $\mathcal{C}$ is the connecting component of Γ_B.

Proof. [RSS2; 1.7 and 1.9].

4. MODULES NOT LYING ON SHORT CYCLES

We say that two modules M and N from mod A have the same composition factors if, for any indecomposable projective A-module P, $\mathrm{Hom}_A(P,M)$ and $\mathrm{Hom}_A(P,N)$ have the same lenght as R-modules.

Using the described relation between short cycles and short chains, and an Auslander-Reiten formula from [AR ; 1.4], one can prove the following

THEOREM 4.1. Let M and N be two indecomposable A-modules with the same composition factors. Assume that M does not lie on a short cycle. Then M and N are isomorphic.

Proof. [RSS1 ; 2.2].

We shall present now some results on the supports of indecomposable modules not lying on short cycles.

PROPOSITION 4.2. Let M be an indecomposable A-module which does not lie on a short cycle in mod A. Then supp M = A/ann M , where ann M is the annihilator of M in A. In particular, if M is sincere, then M is faithful.

Proof. [RSS1 ;Section 3].

THEOREM 4.3. Assume that A admits a sincere indecomposable module which does not lie on a short cycle in mod A. Then

(i) The ordinary quiver of A has no oriented cycles.

(ii) gl.dim.A $\leq$ 2.

(iii) For any indecomposable A-module X either $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$.

Proof. [RSS1 ;Section 3] .

It would be interesting to know whether under assumptions of the above theorem A is a tilted algebra. This is the case for representation-finite algebras as the following theorem shows

THEOREM 4.4. Assume that A is representation-finite and that there exists a sincere indecomposable A-module M which does not lie on a short cycle in mod A. Then M is directing in mod A, and consequently A is a tilted algebra.

Proof. [RSS2 ; 4.4].

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